

Inverses of triangular matrices and bipartite graphs

R.B. Bapat^a E. Ghorbani^{b,c}

^aIndian Statistical Institute, Delhi Centre, 7 S.J.S.S. Marg,
New Delhi 110 016, India

^bDepartment of Mathematics, K.N. Toosi University of Technology,
P. O. Box 16315-1618, Tehran, Iran

^cSchool of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran

rbb@isid.ac.in e_ghorbani@ipm.ir

Abstract

To a given nonsingular triangular matrix A with entries from a ring, we associate a weighted bipartite graph $G(A)$ and give a combinatorial description of the inverse of A by employing paths in $G(A)$. Under a certain condition, nonsingular triangular matrices A such that A and A^{-1} have the same zero-nonzero pattern are characterized. A combinatorial construction is given to construct outer inverses of the adjacency matrix of a weighted tree.

AMS Classification: 05C50, 15A09, 05C05

Keywords: Triangular matrix, bipartite graph, perfect matching, tree, outer inverse

1 Introduction

Let A be a lower triangular matrix with entries from a ring, which is not necessarily commutative. In the first section of this paper we obtain a combinatorial formula for A^{-1} , when it exists. The formula is in terms of certain

paths in the bipartite graph associated with A . We note some consequences of this formula which include expressions for the inverse of a block triangular matrix and a formula for the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching.

In Section 3 we consider lower triangular, invertible, nonnegative matrices A and characterize those such that A and A^{-1} have the same zero-nonzero pattern. This relates to a question posed by Godsil [5] for bipartite graphs. In the final section we provide a combinatorial construction of outer inverses of the adjacency matrix of a weighted tree.

2 Inverses of triangular matrices

Let G be a bipartite graph and let \mathcal{M} be a matching in G . We assume that each edge e of G has a nonzero weight $w(e)$ from a ring (not necessarily commutative). A path in G is said to be *alternating* if the edges are alternately in \mathcal{M} and \mathcal{M}^c , with the first and the last edges being in \mathcal{M} . A path with only one edge, the edge being in \mathcal{M} , is alternating. Let P be the alternating path consisting of the edges e_1, e_2, \dots, e_k in that order. The *weight* $w(P)$ of P is defined to be $w(e_1)^{-1}w(e_2)w(e_3)^{-1} \cdots w(e_{k-1})w(e_k)^{-1}$, assuming that the inverses exist. Thus, if the weights commute, then $w(P)$ is just the product of the weights of the edges in $P \cap \mathcal{M}^c$ divided by the product of the weights of the edges in $P \cap \mathcal{M}$. The length $\ell(P)$ of P is the number of edges on that. For an alternating path P , we define

$$\epsilon(P) = (-1)^{(\ell(P)-1)/2}.$$

Let A be an $n \times n$ matrix with entries from a ring. We associate a bipartite graph $G(A)$ with A as usual: the vertex set is $\{R_1, \dots, R_n\} \cup \{C_1, \dots, C_n\}$ and there is an edge e between R_i to C_j if and only if $a_{ij} \neq 0$, in which case we assign e the weight $w(e) = a_{ij}$. We write vectors as row vectors. The transpose of \mathbf{x} is denoted \mathbf{x}^\top .

Theorem 1. *Let A be a lower triangular $n \times n$ matrix with invertible diagonal elements and \mathcal{M} be the unique perfect matching in $G(A)$ consisting of the edges from R_i to C_i , $i = 1, \dots, n$. Then the entries of $B = A^{-1}$, for $1 \leq j \leq i \leq n$, are given by*

$$b_{ij} = \sum_{P \in \mathcal{P}_{ij}} \epsilon(P)w(P), \tag{1}$$

where \mathcal{P}_{ij} is the set of alternating paths from C_i to R_j in $G(A)$.

Proof. We prove the result by induction on n , the cases $n = 1, 2$ being easy. Assume the result for matrices of order less than n . Partition A and B as

$$A = \begin{pmatrix} A_{11} & \mathbf{0}^\top \\ \mathbf{x} & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & \mathbf{0}^\top \\ \mathbf{y} & b_{nn} \end{pmatrix}.$$

Note that $b_{nn} = a_{nn}^{-1}$ and $B_{11} = A_{11}^{-1}$.

By the induction assumption, (1) holds for $1 \leq j \leq i \leq n-1$. Thus we need to verify (1) for the pairs $(n, 1), \dots, (n, n-1)$.

From $BA = I$ we see that $\mathbf{y}A_{11} + b_{nn}\mathbf{x} = \mathbf{0}$ and hence $\mathbf{y} = -a_{nn}^{-1}\mathbf{x}A_{11}^{-1}$. Therefore

$$y_j = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i b_{ij}, \quad j = 1, \dots, n-1. \quad (2)$$

Consider any alternating path from C_n to R_j in $G(A)$. Any such path must be composed of the edge from C_n to R_n , followed by an edge from R_n to C_i for some $i \in \{1, \dots, n-1\}$, and then an alternating path from C_i to R_j .

If P is an alternating path from C_i to R_j , then denote by P' the alternating path from C_n to R_j obtained by concatenating the edge from C_n to R_n , then the edge from R_n to C_i , followed by P . Note that

$$\epsilon(P')w(P') = -\epsilon(P)a_{nn}^{-1}x_iw(P). \quad (3)$$

By the induction assumption, $b_{ij} = \sum \epsilon(P)w(P)$, where the summation is over all alternating paths from C_i to R_j . Hence it follows from (2) and (3) that for $j = 1, \dots, n-1$,

$$b_{nj} = y_j = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i b_{ij} = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i \left(\sum_{P \in \mathcal{P}_{ij}} \epsilon(P)w(P) \right) = \sum_{P \in \mathcal{P}_{nj}} \epsilon(P)w(P),$$

completing the proof. \square

We note some consequences of Theorem 1. Since the weights are noncommutative, we may take the weights to be square matrices of a fixed order.

This leads to combinatorial formulas for inverses of block triangular matrices. For example, the usual formula

$$\begin{pmatrix} A & O \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}$$

is a consequence of Theorem 1. Another example is the identity

$$\begin{pmatrix} A & O & O & O \\ W & B & O & O \\ X & O & C & O \\ O & Y & Z & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O & O & O \\ -B^{-1}WA^{-1} & B^{-1} & O & O \\ -C^{-1}XA^{-1} & O & C^{-1} & O \\ D^{-1}YB^{-1}WA^{-1} + D^{-1}ZC^{-1}XA^{-1} & -D^{-1}YB^{-1} & -D^{-1}ZC^{-1} & D^{-1} \end{pmatrix}.$$

We note yet another consequence of Theorem 1. Let $\text{GF}(2)$ denote the Galois field of order 2. The following result easily follows from Theorem 1.

Corollary 2. *Let A be an $n \times n$ lower triangular matrix over $\text{GF}(2)$ such that $a_{ii} = 1$, $i = 1, \dots, n$; and let $B = A^{-1}$. Let $G(A)$ be the graph associated with A . Then $b_{ij} = 1$ if and only if there are an odd number of alternating paths from C_i to R_j in $G(A)$.*

If A is a lower triangular matrix, then

$$\begin{pmatrix} O & A \\ A^\top & O \end{pmatrix} \tag{4}$$

is the (weighted) adjacency matrix of a bipartite graph with a unique perfect matching. Conversely the adjacency matrix of a bipartite graph with a unique perfect matching can be put in the form (4) after a relabeling of the vertices. In view of this observation, the unweighted case of Theorem 1 can be seen to be equivalent to Lemma 2.1 of Barik, Neumann and Pati [2]. Our proof technique is different. In the same spirit, Theorem 1 leads to a formula for the inverse of the adjacency matrix of a weighted tree (see Section 4) when the tree has a perfect matching, generalizing a well-known result from [4, 7] (see also [1, Section 3.6]).

Remark 3. Let T be tree with nonsingular weighted adjacency matrix A . Then A^{-1} is the weighted adjacency matrix of a bipartite graph. The graphs that can occur as inverses of nonsingular trees were characterized in [6]. Namely, a graph G is the inverse of some tree if and only if $G \in \mathcal{F}_k$ where \mathcal{F}_k is the family of graphs defined recursively as follows. Set $\mathcal{F}_1 = \{P_2\}$ and for $k \geq 2$ any $G \in \mathcal{F}_k$ is obtained from some $H \in \mathcal{F}_{k-1}$ by taking any vertex u of H and adding two new vertices u' and v where u' is joined to all the neighbors of u and v (a pendant vertex) is joined to u' . The characterization remains valid in the more general setting when the weights of the edges come from a ring (provided the required inverses of the weights exist).

3 Matrices with isomorphic inverses

In this section we consider real matrices. It is an interesting problem to determine the triangular matrices A for which $G(A)$ is isomorphic to $G(A^{-1})$. This problem is in close connection with the one posed by Godsil [5] as described below.

Let G be a bipartite graph on $2n$ vertices which has a unique perfect matching \mathcal{M} . Then there is a lower triangular matrix A such that $G = G(A)$. With the additional hypothesis that the graph G/\mathcal{M} , obtained from G by contracting the edges in \mathcal{M} , is bipartite, Godsil [5] showed that A^{-1} is diagonally similar to a matrix A^+ whose entries are nonnegative and which dominates A , that is $A^+(i, j) \geq A(i, j)$ for all $1 \leq i, j \leq n$. In turn, A^+ can be regarded as the adjacency matrix of a bipartite multigraph G^+ in which G appears as a subgraph. In this framework, Godsil asked for a characterization of the graphs G such that G^+ is isomorphic to G . This was answered in [8], by showing that G and G^+ are isomorphic if and only if G is a corona of bipartite graph. The *corona* of a graph is obtained by creating a new vertex v' for each vertex v such that v' is adjacent to v . The following theorem is a generalization of this result.

Theorem 4. *Let A be a lower triangular matrix with nonnegative entries, \mathcal{M} being the unique matching of $G = G(A)$ and such that G/\mathcal{M} is bipartite. Then A and A^{-1} have the same zero-nonzero pattern if and only if G is a corona of a bipartite graph.*

Proof. If G is a corona, by some rearranging, we may write A as

$$A = \begin{pmatrix} I & O \\ A_0 & I \end{pmatrix},$$

for some A_0 . Hence

$$A^{-1} = \begin{pmatrix} I & O \\ -A_0 & I \end{pmatrix},$$

proving the ‘if’ part of the theorem.

Next, assume that A and A^{-1} have the same zero-nonzero pattern. To show that G is a corona, it suffices to prove that the alternating paths of G are of length at most 3. By contradiction, suppose that G has an alternating path of length larger than 3 and so it has an alternating path of length 5 between R_j and C_i , say. Since G/\mathcal{M} is bipartite, all the alternating paths between R_j and C_i must have the same length mod 4 (note that two alternating paths with different lengths mod 4 between two vertices give rise to an odd cycle in G/\mathcal{M}). So, by Theorem 1, the (i, j) entry of A^{-1} is nonzero. Since A and A^{-1} have the same zero-nonzero pattern, the (i, j) entry of A is nonzero and hence R_j and C_i are adjacent. This implies the existence of a triangle in G/\mathcal{M} , a contradiction. \square

4 Generalized inverses and matchings

Let A be an $m \times n$ matrix with entries from a ring such that $T = G(A)$ is a tree and let \mathcal{M} be a matching in T . When \mathcal{M} is perfect, A is nonsingular and a formula for A^{-1} may be given in terms of alternating paths, as noted at the end of Section 2. When \mathcal{M} is not perfect, we still may define an $n \times m$ matrix $B = (b_{ij})$ using the alternating paths of \mathcal{M} in the same fashion as when \mathcal{M} is a perfect matching. More precisely, if $\{R_1, \dots, R_m\}$ and $\{C_1, \dots, C_n\}$ are color classes of T , then for $1 \leq j \leq i \leq n$,

$$b_{ij} = \sum \epsilon(P)w(P),$$

where the summation is over all alternating paths P from C_i to R_j in $G(A)$. We call such a matrix the *path matrix of T with respect to \mathcal{M}* . We show that the path matrix turns out to be an outer inverse of the adjacency matrix.

Theorem 5. *Let A be an $m \times n$ matrix such that $T = G(A)$ is a tree and let \mathcal{M}_1 and \mathcal{M}_2 be two matchings in T with $\mathcal{M}_2 \subseteq \mathcal{M}_1$. Let B_1 and B_2 be $n \times m$ path matrices of T with respect to \mathcal{M}_1 and \mathcal{M}_2 , respectively. Then*

$$B_1AB_2 = B_2AB_1 = B_2.$$

Proof. Let F_1 and F_2 be the induced forests by T on the vertices saturated by \mathcal{M}_1 and \mathcal{M}_2 , respectively. Let A_1 and A_2 be the submatrices of A such that $F_1 = G(A_1)$ and $F_2 = G(A_2)$. Then \mathcal{M}_1 and \mathcal{M}_2 are perfect matchings for F_1 and F_2 , respectively. Let $|\mathcal{M}_1| = p$ and $|\mathcal{M}_2| = q$. It turns out that, with an appropriate ordering of the vertices,

$$B_1 = \begin{pmatrix} A_1^{-1} & O_{p \times (m-p)} \\ O_{(n-p) \times p} & O_{(n-p) \times (m-p)} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} A_2^{-1} & O_{q \times (m-q)} \\ O_{(n-q) \times q} & O_{(n-q) \times (m-q)} \end{pmatrix}.$$

Note that A_2^{-1} is also a submatrix of A_1^{-1} , so B_1 is in fact of the form

$$B_1 = \left(\begin{array}{c|c|c} A_2^{-1} & O & O \\ \hline * & * & O \\ \hline O & O & O \end{array} \right).$$

Then

$$AB_1 = \begin{pmatrix} I_{p \times p} & O_{p \times (m-p)} \\ * & O_{(m-p) \times (m-p)} \end{pmatrix}.$$

It follows that

$$B_2AB_1 = \left(\begin{array}{c|c|c} A_2^{-1} & O & O \\ \hline O & O & O \\ \hline O & O & O \end{array} \right) = B_2.$$

The equality $B_1AB_2 = B_2$ is proved similarly. \square

With the same proof as the theorem above, we can prove even a more general statement as follows.

Theorem 6. *Let A be an $m \times n$ matrix such that $T = G(A)$ is a tree and let \mathcal{M}_1 and \mathcal{M}_2 be two matchings in T . If B_1 and B_2 be $n \times m$ path matrices of T with respect to \mathcal{M}_1 and \mathcal{M}_2 , respectively, then*

$$B_1AB_2 = B_2AB_1 = C,$$

where C is the path matrix of T with respect to $\mathcal{M}_1 \cap \mathcal{M}_2$.

Recall that the matrix B is called a *2-inverse* (or an *outer inverse*) of the matrix A if $BAB = B$ (see, for example, [3]). The next result is an immediate consequence of Theorem 5.

Corollary 7. *Let A be a matrix such that $T = G(A)$ is a tree and let \mathcal{M} be a matching in T . If B is the path matrix of T with respect to \mathcal{M} , then B is an outer inverse of A .*

Acknowledgments

The first author acknowledges support from the JC Bose Fellowship, Department of Science and Technology, Government of India, and also thanks IPM, Tehran, for hospitality during a visit when this research was carried out. The research of the second author was in part supported by a grant from IPM (No. 91050114).

References

- [1] R.B. Bapat, *Graphs and Matrices*, Springer, London; Hindustan Book Agency, New Delhi, 2010.
- [2] S. Barik, M. Neumann, and S. Pati, On nonsingular trees and a reciprocal eigenvalue property, *Linear Multilinear Algebra* **54** (2006), 453–465.
- [3] A. Ben-Israel and T.N.E. Greville, *Generalized Inverses: Theory and Applications*, Second Edition, Springer, New York, 2003.
- [4] F. Buckley, L.L. Doty, and F. Harary, On graphs with signed inverses, *Networks* **18** (1998), 151–157.
- [5] C.D. Godsil, Inverses of trees, *Combinatorica* **5** (1985), 33–39.
- [6] M. Neumann and S. Pati, On reciprocal eigenvalue property of weighted trees, *Linear Algebra Appl.* (2011) doi:10.1016/j.laa.2011.09.017
- [7] S. Pavlikova and J. Krc-Jediny, On the inverse and dual index of a tree, *Linear Multilinear Algebra* **28** (1990), 93–109.

- [8] R. Simion and D.-S. Cao, Solution to a problem of C.D. Godsil regarding bipartite graphs with unique perfect matching, *Combinatorica* **9** (1989), 85–89.